SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 3

Problem 2. Let S_1 and S_2 be level sets of functions $F_1, F_2 : \mathbb{R}^3 \to \mathbb{R}$ at regular values, respectively. Show that $S_1 \cap S_2$ is the level set of a function F (you have to find it), and use this function to show that if for every $x \in S_1 \cap S_2$, $\ker dF_1(x) \neq \ker dF_2(x)$, then $S_1 \cap S_2$ is a 1-manifold.

Problem 3. Let $S(x, r)$ denote the sphere of radius r based a a point $x \in \mathbb{R}^3$, and $C(v, r)$ denote the cylinder centered at the line through 0 in the unit vector v of radius r:

$$C(v, r) = \left\{ y \in \mathbb{R}^3 : ||\pi_v(y)|| = r \right\},$$

where $\pi_v : \mathbb{R}^3 \to \langle v \rangle^\perp$ is the orthogonal projection onto the orthogonal complement of v.

(a) Find functions $F_{v,r} : \mathbb{R}^3 \to \mathbb{R}$ and $G_{x,r} : \mathbb{R}^3 \to \mathbb{R}$ such that $C(v, r)$ and $S(x, r)$ are levels sets of F and G respectively.

(b) For which points, vectors and radii are the sphere and cylinder $S(x, r_1)$ and $C(v, r_2)$ transverse? For which are they nontrivially transverse?

Remark 1. A submersion with compact preimages always induces a "local product structure" between open subsets of the base manifold N and the fibers $F^{-1}(y)$ as described by the previous problem. Such structures are also sometimes called *fiber bundles* or *fibrations*.

Problem 4. Let $M \subset \mathbb{R}^3$ be a compact surface and $E \subset \mathbb{R}^3$ be a plane passing through 0. Show that there exists a plane $E + v$ parallel to E such that $M \cap (E + v)$ is a nonempty union of circles. [*Hint*: If you want to apply Sard's theorem, make sure you read the fine print (ie, pay attention to the difference between a regular value and a non-trivial regular value). The conclusion fails when M is not compact!]